A Simple Model for the Dynamic Behavior of the Circulatory System

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OUTLINE

Motivation (Why is the human heart rate around 1Hz?)

Dynamic Permeability of Viscoelastic Fluids in a tube

Enhancement in the Dynamic Permeability

Bethe lattice for modeling the Circulatory System

Results (shift to higher frequencies and lesser enhancement)

Future work

Some facts

Human heart rate ranges from 0.9 to 3 Hz

Musaraña heart rate is around 10 Hz

Blue whale heart rate is around 0.5 Hz

Why?

Dynamic permability of Maxwellian fluid in a tube

Continuity equation for incompressible flow,

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

and the linearized momentum equation

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p - \nabla \cdot \boldsymbol{\tau} \ , \tag{2}$$

in the absence of any non-uniform external field. Here p represents the pressure that may contain the effect of gravity, $\mathbf{v}$ is the fluid velocity, ρ is the mass density of the fluid and $\boldsymbol{\tau}$ represents the viscous stress tensor.

To study a viscoelastic fluid we consider the linear form of the Maxwell model

$$t_m \frac{\partial \boldsymbol{\tau}}{\partial t} = -\eta \nabla \mathbf{v} - \boldsymbol{\tau}, \tag{3}$$

Fourier transform

$$\rho \left(t_m \omega^2 + i\omega \right) \mathbf{V} + \eta \nabla^2 \mathbf{V} = \left(1 - i\omega t_m \right) \nabla P. \tag{4}$$

The solution to Bessel's equation with no-slip condition $V(a) = 0$ is given by

$$V(r) = -\frac{(1 - i\omega t_m)}{\beta^2}\left[1 + \frac{J_0(\beta r)}{J_0(\beta a)}\right]\frac{\partial P}{\partial z},$$

where β

$$\beta = \left(\frac{\rho}{\eta t_m}\left[(t_m\omega)^2 + i\omega t_m\right]\right)^{\frac{1}{2}}.$$

Average velocity denoted by $< V >$

$$< V >= -K(\omega)\frac{\partial P}{\partial z}.$$

Here the dynamic permeability is given by

$$K(\omega) = -\frac{a^2\,(1 - i\omega t_m)}{\alpha\varpi}\left[1 - \frac{2}{\sqrt{\alpha\varpi}J_0(\sqrt{\alpha\varpi})}J_1(\sqrt{\alpha\varpi})\right],$$

in which α^{-1} is the Deborah number, $\alpha = \rho a^2/\eta t_m$, and the parameter ϖ is defined by

$$\varpi(\omega) = (\omega^*)^2 + i\omega^*$$

with the dimensionless frequency ω^* given by $\omega^* = t_m\omega$

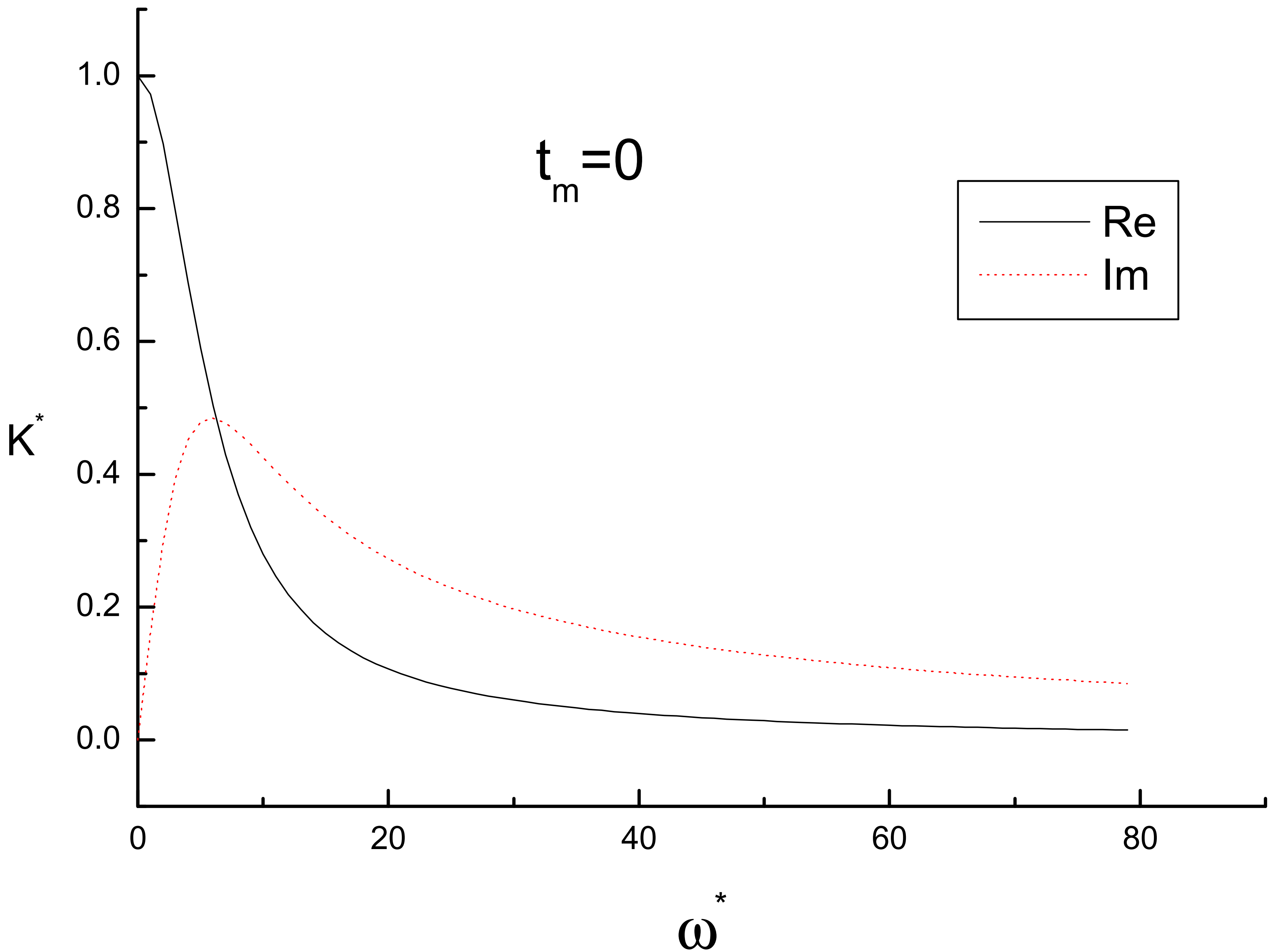
In the limit $t_m \to 0$ we recover the result of Zhou and Sheng.

Dimensionless Permeability

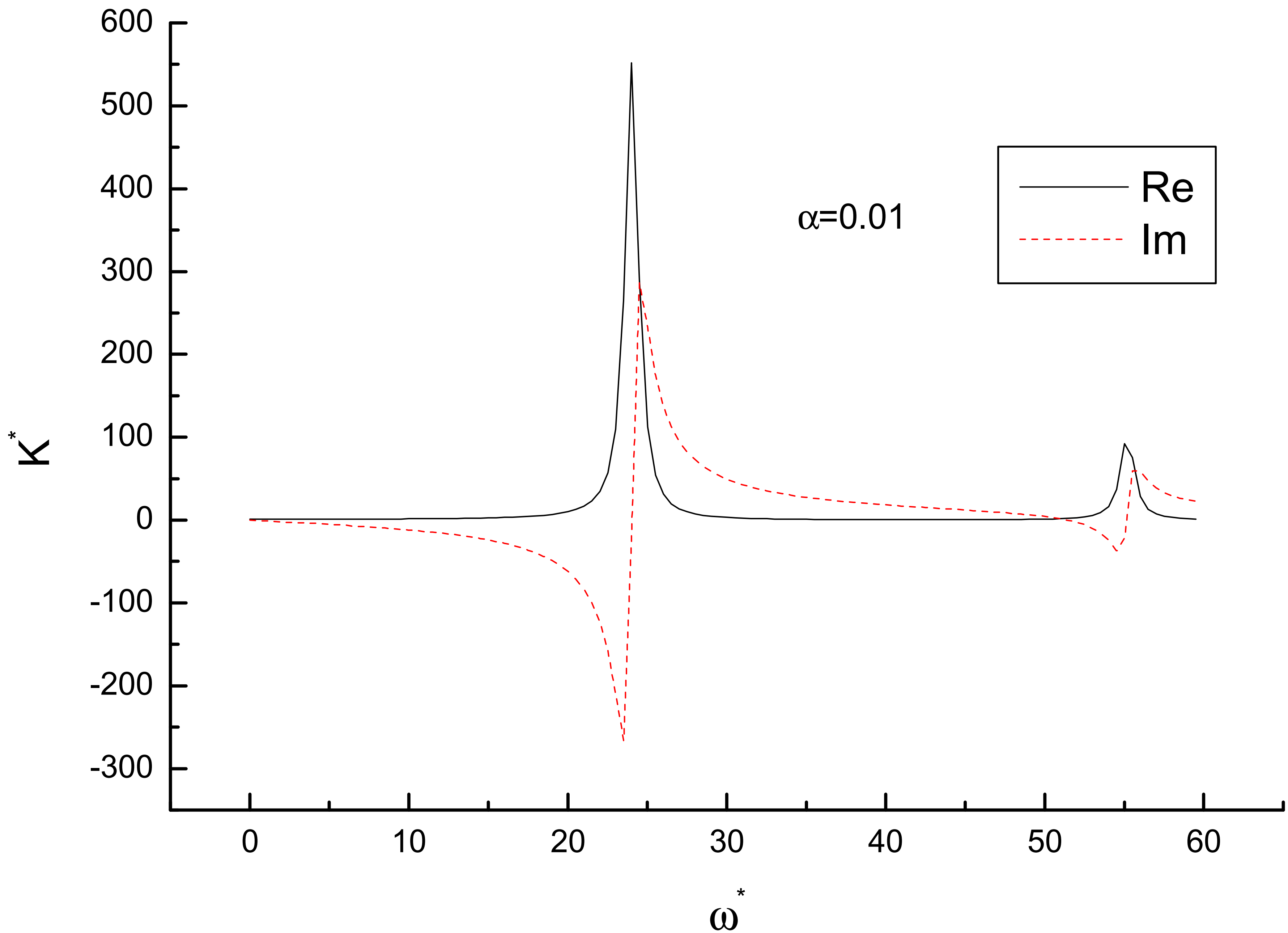
The dimensionless permeability with respect to the steady state is

$$K^*(\omega) = \frac{8\,(1 - it_m)}{\alpha\varpi}\left[1 - \frac{2}{\sqrt{\alpha\varpi}J_0(\sqrt{\alpha\varpi})}J_1(\sqrt{\alpha\varpi})\right].$$

This is a result for a single tube o for a bundle of capillary tubes (López de Haro et al. TIPM 1996 and del Río et al. PRE 1998).



1.0
0.8
0.6
0.4
0.2
0.0
K*
t_m=0
Re
Im
0
20
40
60
80
ω*



600
500
400
300
200
100
0
-100
-200
-300
0
10
20
30
40
50
60
K*
ω*
α=0.01
Re
Im

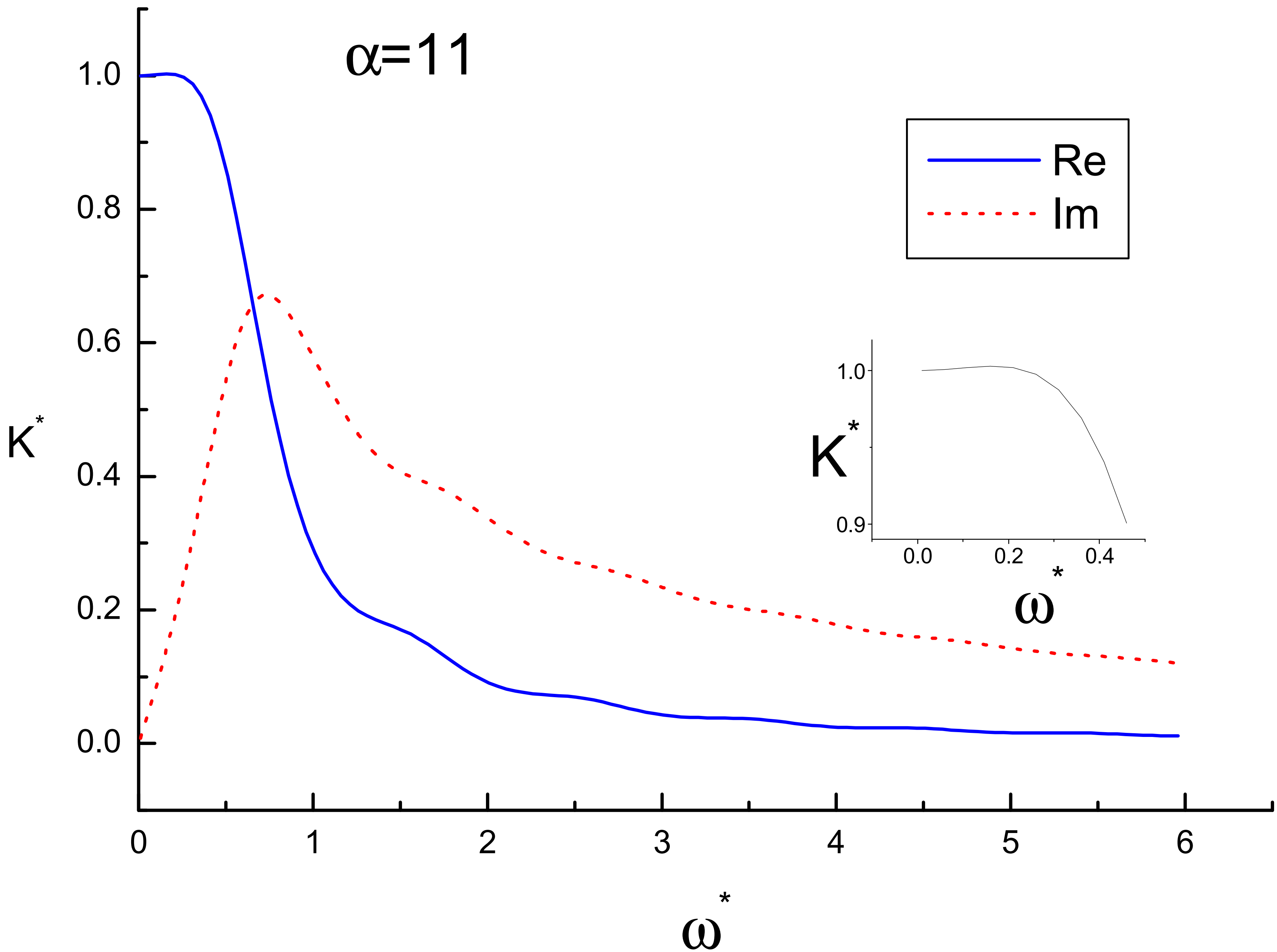
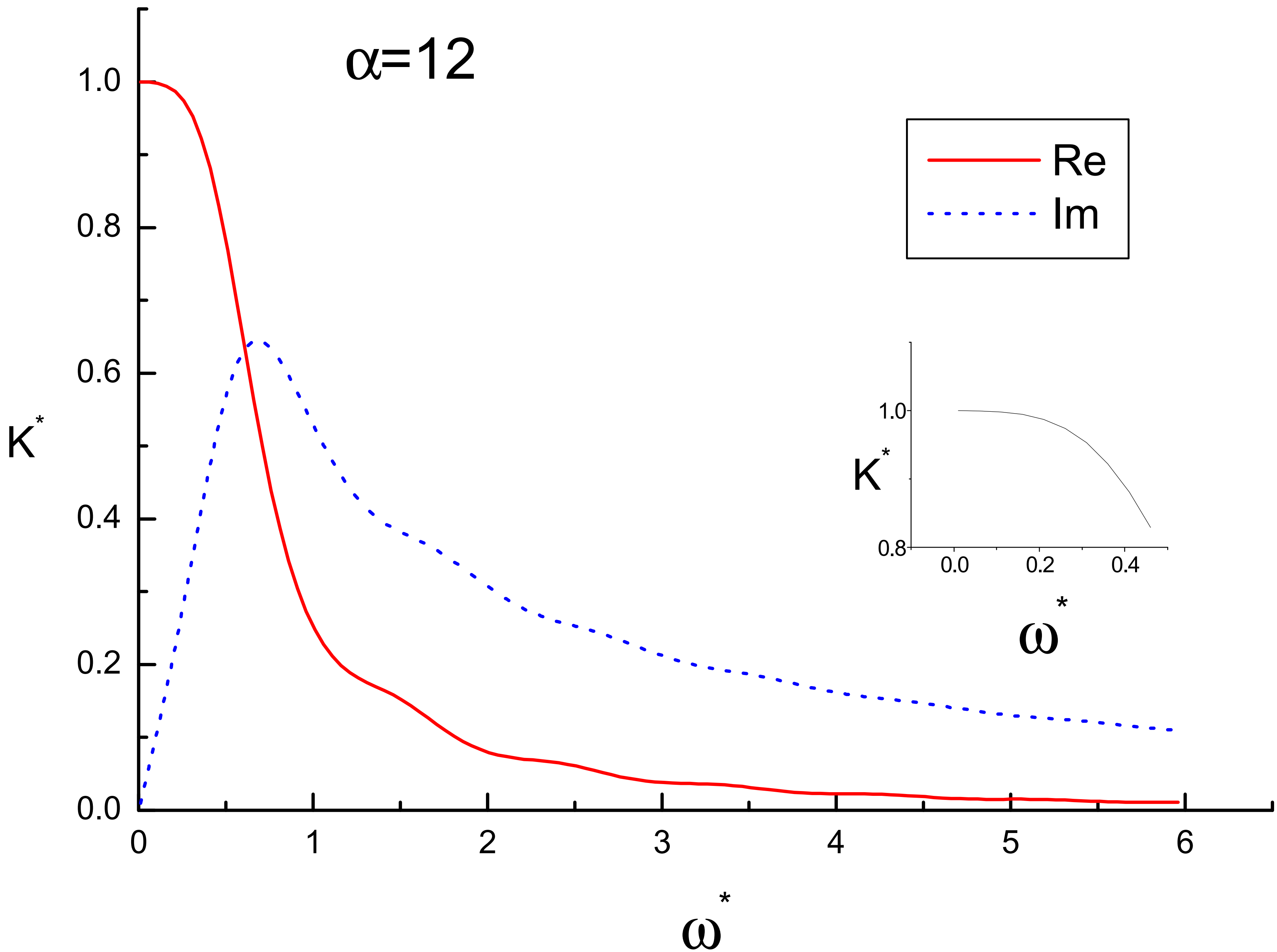
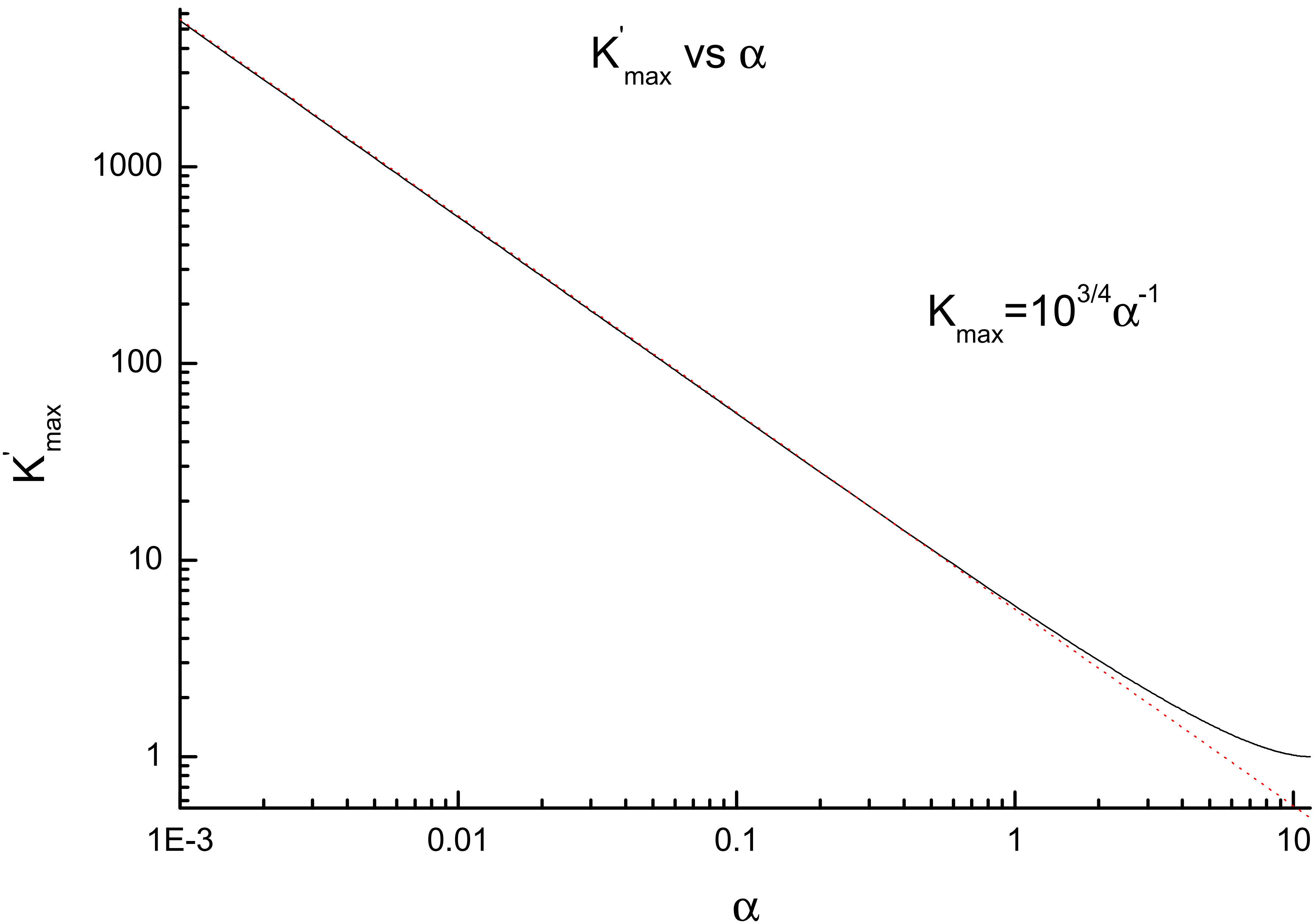
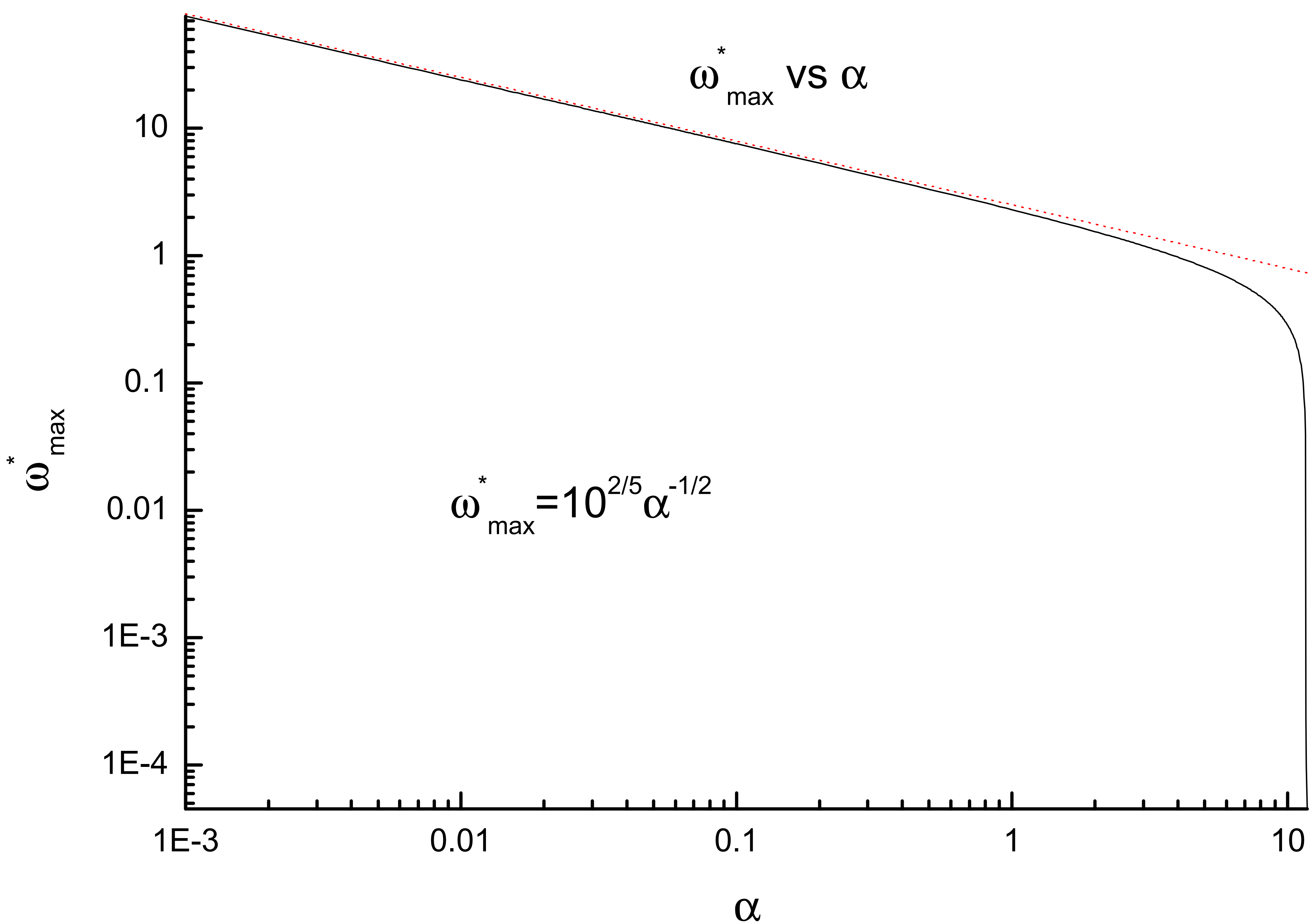

α=11
1.0
0.8
0.6
0.4
0.2
0.0
K*
0
1
2
3
4
5
6
ω*
Re
Im
1.0
0.9
K*
0.0
0.2
0.4
ω*

fig. 5





K'max vs α
Kmax=10^3/4α^-1
1000
100
10
1
K'max
1E-3
0.01
0.1
1
10
α



ω^*_{max} vs α
$\omega^*_{max}=10^{2/5}\alpha^{-1/2}$
10
1
0.1
0.01
1E-3
1E-4
1E-3
0.01
0.1
1
10
α
ω^*_{max}

Simple relationships

$$K^*_{\max} = \frac{10^{\frac{3}{4}}}{\alpha}$$

and

$$\omega^*_{\max} = \frac{10^{\frac{2}{5}}}{\sqrt{\alpha}}$$

combination of these two results leads to

$$K^*_{\max} = 10^{\frac{1}{5}} \left(\omega^*_{\max}\right)^2$$

This expression indicates that, in the elastic regime, the maximum value for the averaged velocity grows more rapidly than the frequency of the oscillating pressure required to reach such maximum. The transition between viscous and elastic behavior occurs at $\alpha = 11.64$.

What is the optimum human heart rate?

Here the dynamic permeability is given by

$$K(\omega) = -\frac{a^2\,(1 - i\omega t_m)}{\alpha\varpi}\left[1 - \frac{2}{\sqrt{\alpha\varpi}J_0(\sqrt{\alpha\varpi})}J_1(\sqrt{\alpha\varpi})\right],$$

$$\alpha = \rho a^2/\eta t_m,$$

Human heart rate

$r \sim 0.02$ to $0.35cm$, $\rho \sim 1.06g/cm^3$,
$\eta \sim 0.05$ to $0.08p$, and $t_m \sim 1s$

(Thruston, Biorehology 1996)

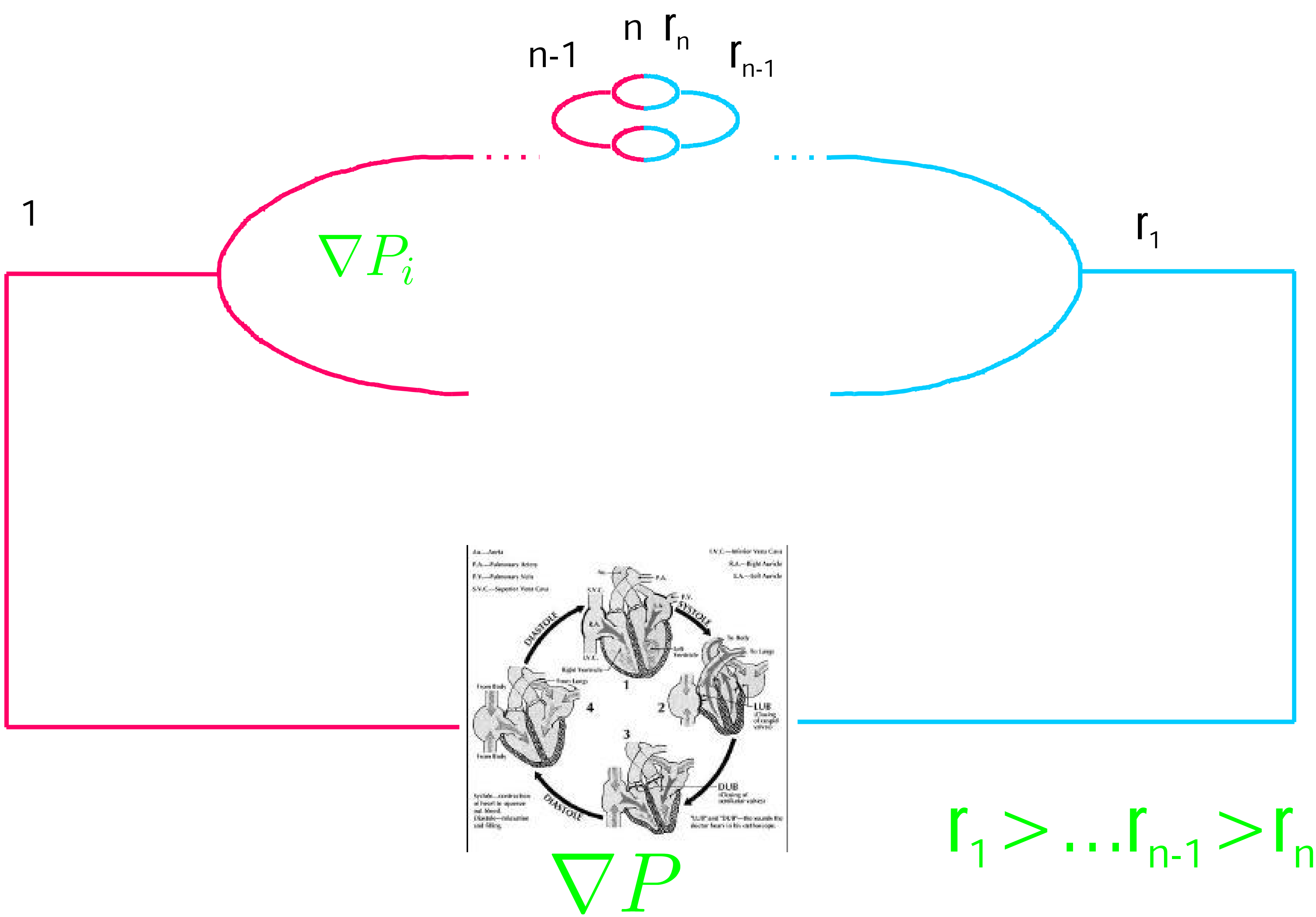
Optimum human heart rate

0.25 Hz to 8.7 Hz

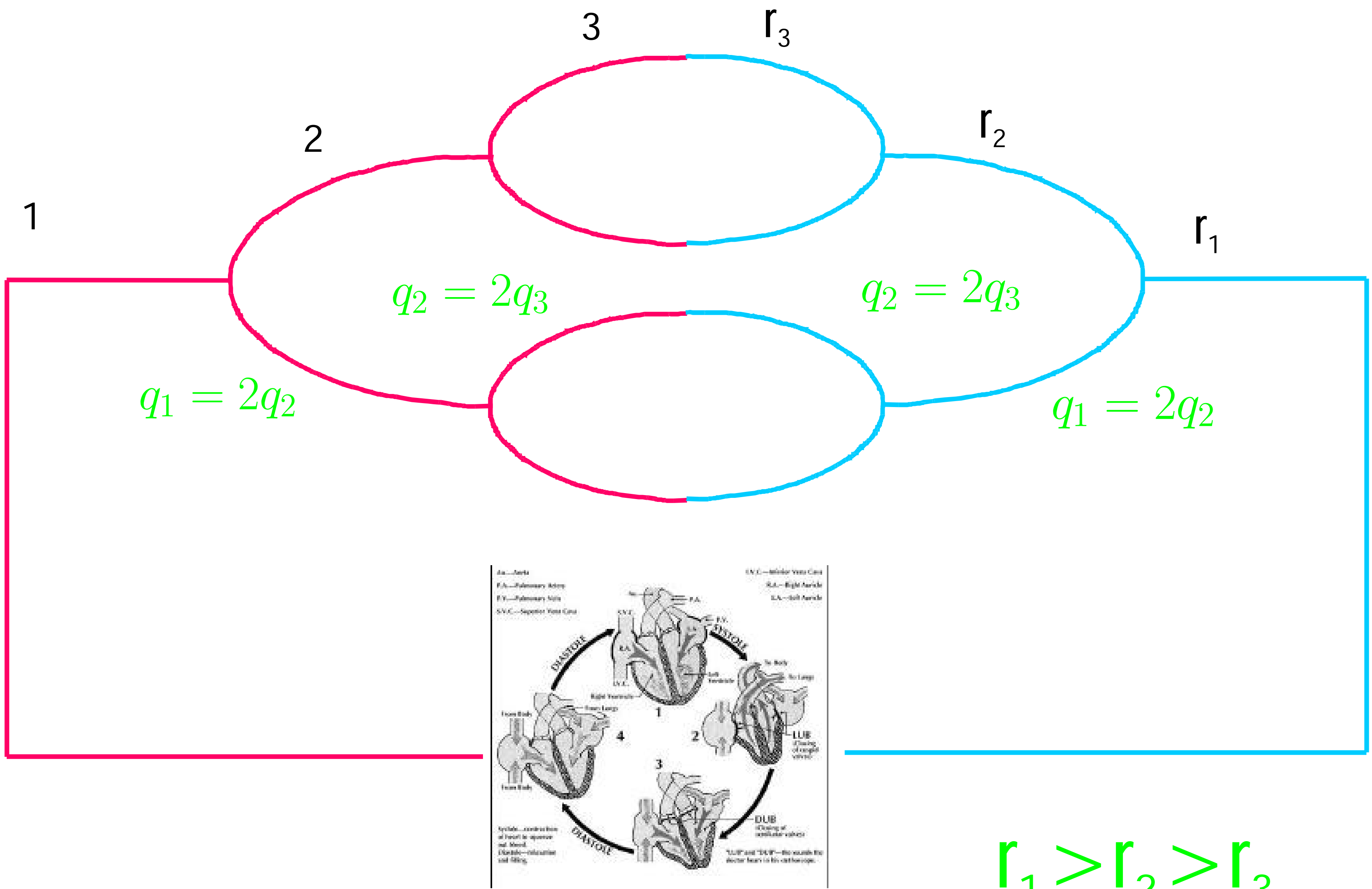
(del Río et al. PRE 1998, 2001)

For a single tube

n step Bethe lattice



3 step Bethe lattice



$r_1 > r_2 > r_3$

The total averaged flow $q =< V >$ passes through the first tube and then it is divided in two, then in four and so on. The emerging system of equations is given by

$$q_1 = 2q_2$$

$$q_2 = 2q_3$$

and

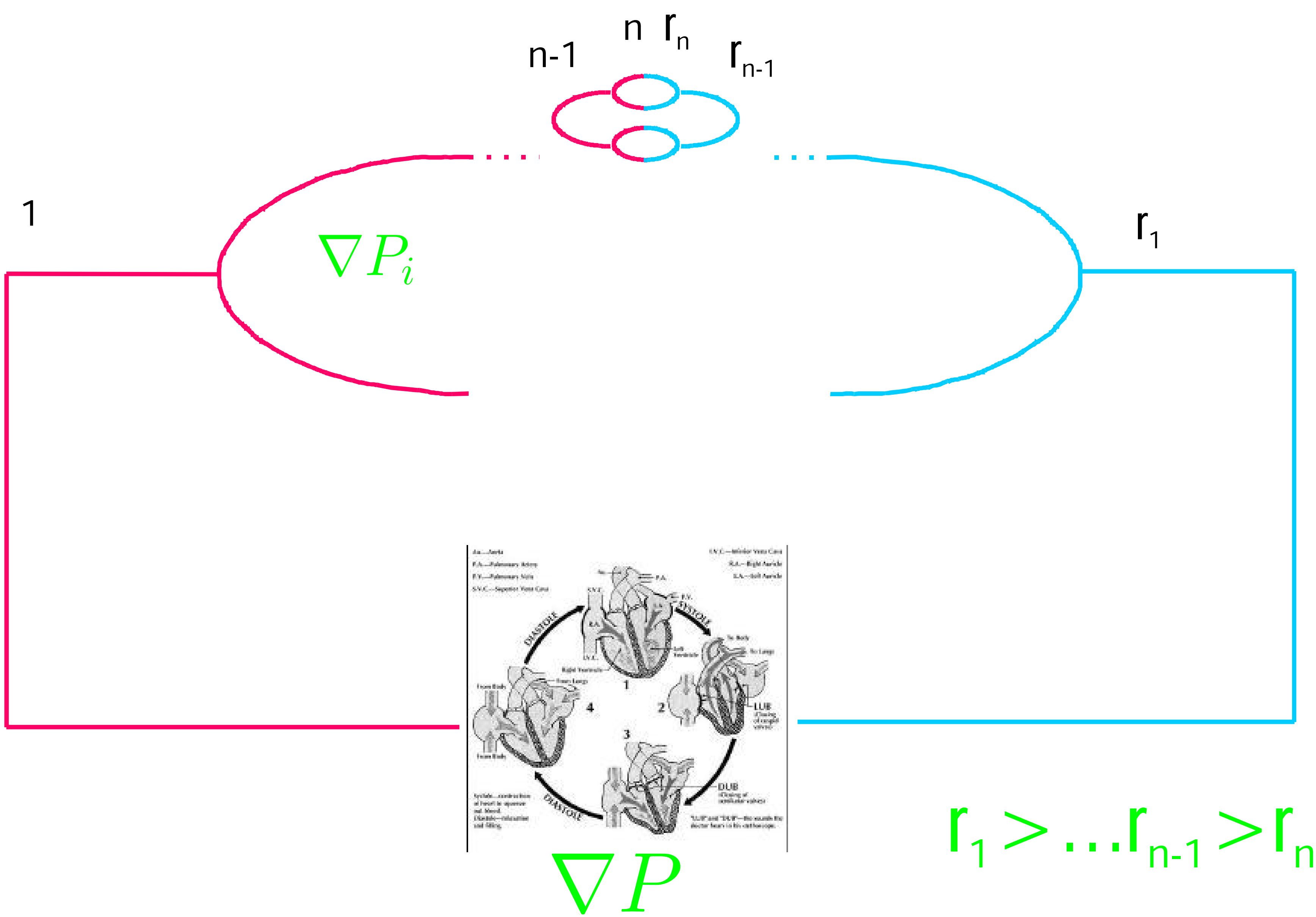
$$q_{n-1} = 2q_n$$

with

$$q_i = -k_i \nabla P_i$$

where ∇P_i is the pressure drop in each segment of the lattice. We thus have $2n - 1$ coupled equations.

n step Bethe lattice



We want to obtain the effective permeability. This is given by

$$q_1 = -k_{eff} \nabla P$$

It is important to stress that, with the exception of the last level, all the pressure drops appear two times, and due to the fact that

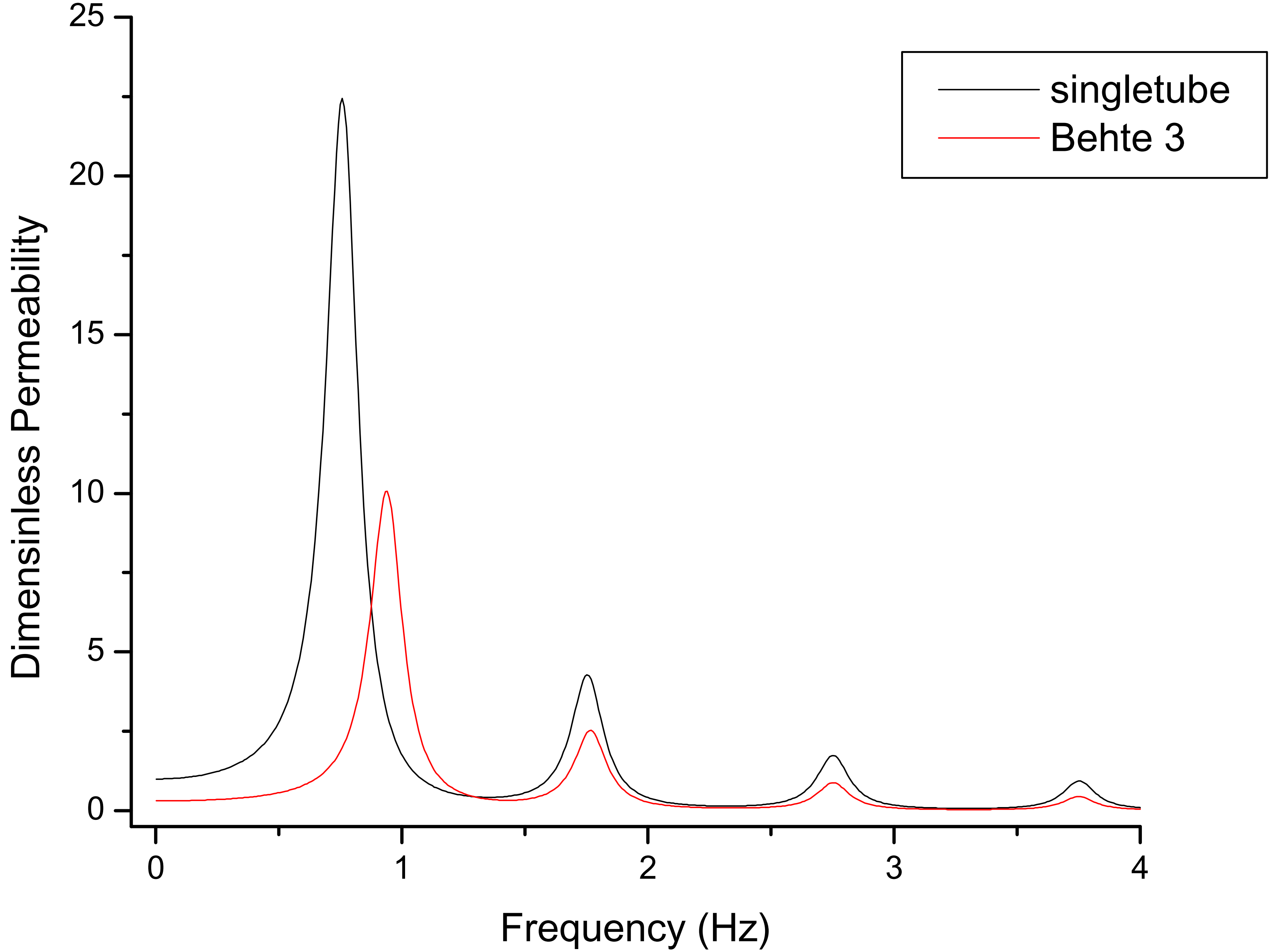
$$\nabla P = \sum_{i=1}^{n} \nabla P_i + \sum_{i=1}^{n-1} \nabla P_i$$

$$\frac{q_1}{k_{eff}} = \sum_{i=1}^{n} \frac{q_i}{k_i} + \sum_{i=1}^{n-1} \frac{q_i}{k_i}$$

but $q_1 = 2^{i-1} q_i$, so that

$$\frac{1}{k_{eff}} = \frac{1}{2^{n-1} k_n} + \sum_{i=1}^{n-1} \frac{2^{2-i}}{k_i}$$

is the efective permeability of the Bethe lattice model for the circulatory system.



Dimensinless Permeability
Frequency (Hz)
singletube
Behte 3
0
5
10
15
20
25
0
1
2
3
4

Concluding Remarks

The viscoelastic nature of blood is an important physical aspect determining the heart rate.

The consideration of a lattice maintains the resonant character of the blood in arteries.

This simple Bethe Lattice model describes qualitatively the dynamic behavior of the circulatory system.

This results seems to explain the scaling laws for heart rate and animal sizes or weigths.